Math 250

6.2 Area of a Region Between Two Curves

Objectives

1) Find the area of a region between two curves when limits of integration are given in the question.

2) Find the area of a region between two curves that intersect using the points of intersection to identify the limits of integration.

3) Reverse the roles of x and y for these two types of area

4) Describe integration as an accumulation process

Key components

- Know which function to write first
- Always subtract the two functions
- Limits of integration are edges of area
- Result is always positive - it is a true area (not a net signed area like we get from a

- dx means we are taking a sum in the x-direction
- dy means we are taking a sum in the y-direction.

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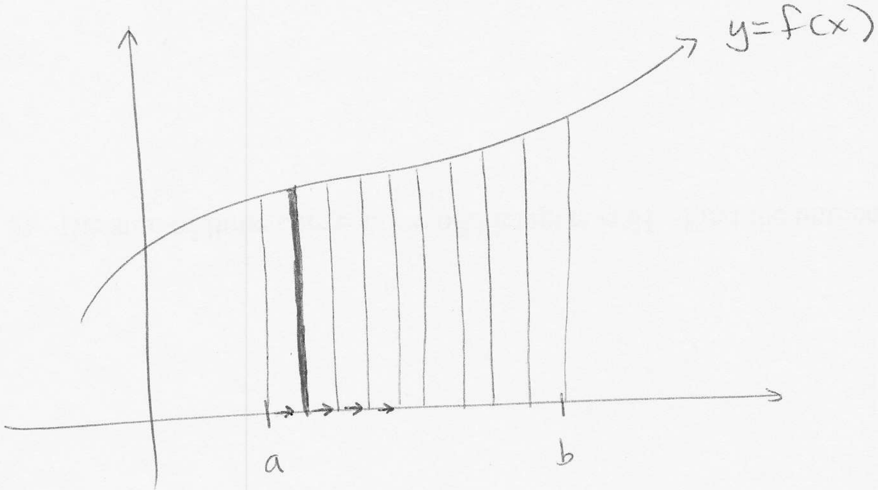
Integration as an accumulation process

To accumulate: to pile up, gather, collect

Accumulation: the stuff that's been gathered, piled, or collected



We will use the idea of accumulation extensively in 7.2 and 7.3 when we calculate volumes.

Recall Riemann sums? $\Rightarrow$ divide the region into many rectangles:

Area of each rectangle	$\Rightarrow$	Use function for height	$\Rightarrow$	Add up all rectangles	$\Rightarrow$	Take limit
$L \cdot W$		$\underbrace{f(x_i^*)}_{L} \cdot \underbrace{\Delta x_i}_{W}$		$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$		$\int_a^b f(x)\,dx$

The integral is an accumulation of areas of infinitesimally small rectangles.

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(1) Find the area of the region bounded by

$y = x^2 + 1$

$y = x$

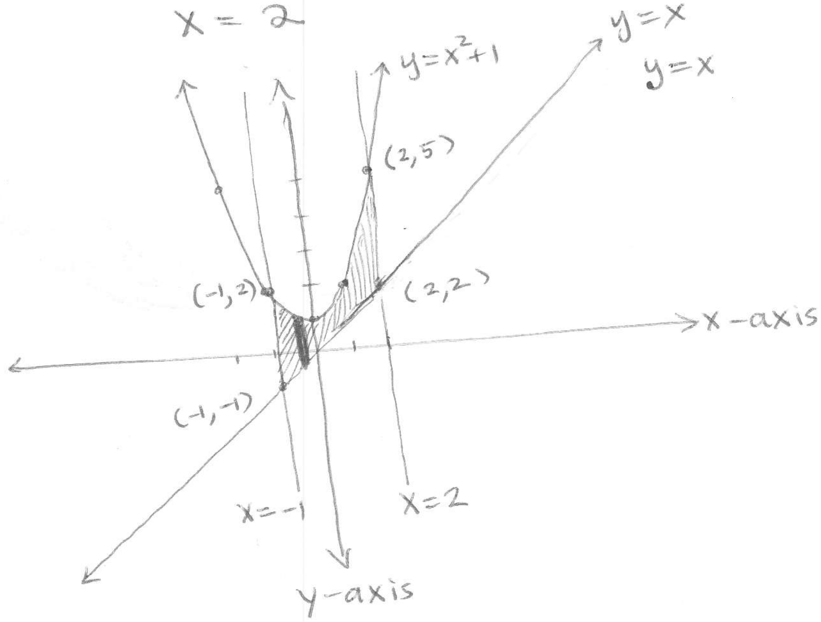
$x = -1$

$x = 2$

step 1

sketch graph

shade region



accumulate the height of the bar from $x = -1$ to $x = 2$

dx $\rightarrow \rightarrow \rightarrow \rightarrow$ x-direction.

means integrate in x.

$$\int_{x=-1}^{x=2} \text{upper} - \text{lower} \, dx$$

$$= \int_{x=-1}^{x=2} (x^2+1) - (x) \, dx$$

$$= \int_{-1}^{2} x^2 + 1 - x \, dx$$

$$= \left(\frac{1}{3}x^3 + x - \frac{1}{2}x^2\right)\Big|_{-1}^{2}$$

$$= \left(\frac{8}{3} + 2 - 2\right) - \left(\frac{-1}{3} - 1 - \frac{1}{2}\right)$$

$$= \boxed{\frac{9}{2}} = \boxed{4.5}$$

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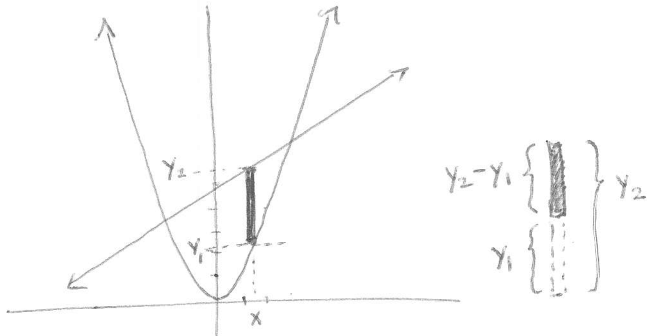
Find the area of the region bounded by

2

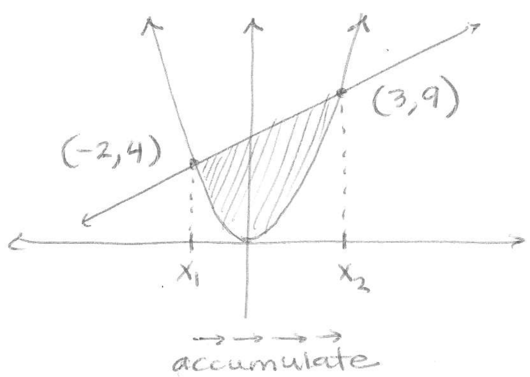
$y_1 = x^2 \longrightarrow (x,y) = (x, x^2) = (\pm\sqrt{y}, y)$ (depends)

$y_2 = x+6 \longrightarrow (x, x+6) = (y-6, y)$



$y_2 - y_1 = x + 6 - x^2$

Always sketch and shade your intended region!



desired area = accumulate all the bars above.

$$\int_{x_1=-2}^{x_2=3} x + 6 - x^2 \, dx$$

Find points of intersection:

$x^2 = x + 6$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3 \quad x = -2$

Antidifferentiate

$$= \left[\tfrac{1}{2}x^2 + 6x - \tfrac{1}{3}x^3\right]\Big|_{-2}^{3}$$

$$= \left(\tfrac{1}{2}(3)^2 + 6(3) - \tfrac{1}{3}(3)^3\right)$$

$$- \left(\tfrac{1}{2}(-2)^2 + 6(-2) - \tfrac{1}{3}(-2)^3\right)$$

$$= \boxed{\frac{125}{6}}$$

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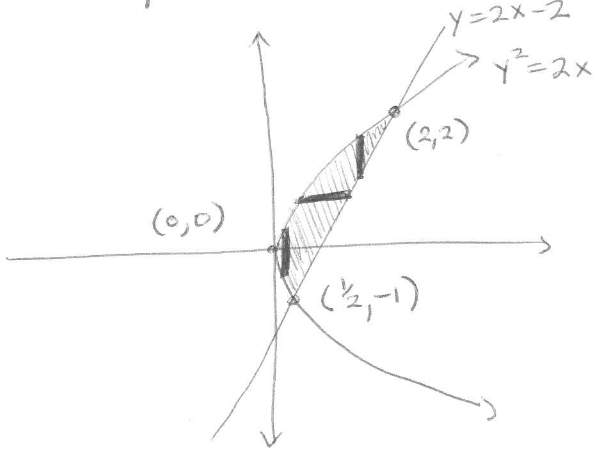
(3) Find the area of the region bounded by

$$\begin{cases} y^2 = 2x & \Rightarrow x = \frac{1}{2}y^2 \text{ right-opening parabola, vertex } (0,0) \text{, wider} \\ y = 2x-2 & \Rightarrow \text{line.} \Longrightarrow \frac{1}{2}(y+2) = x \end{cases}$$

$\Rightarrow y = \pm\sqrt{2x}$



Points of intersection

$2x = y^2$ subst for $2x$

$y = y^2 - 2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y=2$	$y=-1$
$x=2$	$x=\frac{1}{2}$
$(2,2)$	$(\frac{1}{2},-1)$

option 1 : accumulate in x

→ → →

upper parabola y coord minus lower parabola y coord + upper parabola y-coord minus line y-coord

Expressed in terms of x

Limits of integration in x

dx

$$\int_{x=0}^{x=1/2} \sqrt{2x} - (-\sqrt{2x})\,dx + \int_{x=1/2}^{x=2} \sqrt{2x} - (2x-2)\,dx$$

$$= \int_{x=0}^{1/2} 2\sqrt{2}\cdot x^{1/2}\,dx + \int_{1/2}^{2} \sqrt{2}\cdot x^{1/2} - 2x + 2\,dx$$

$$= 2\sqrt{2}\left(\frac{2}{3}x^{3/2}\right)\Big|_0^{1/2} + \left(\sqrt{2}\cdot\frac{2}{3}x^{3/2} - x^2 + 2x\right)\Big|_{1/2}^{2}$$

⋮

much pain

⋮

$$= \boxed{\frac{9}{4}} = \boxed{2.25}$$

option 2 : accumulate in y

line x coord − parabola x-coord

Expressed in terms of y

Limits of integration in y

dy

$$\int_{y=-1}^{y=2} \frac{1}{2}(y+2) - \frac{1}{2}y^2\,dy$$

$$= \frac{1}{2}\int_{-1}^{2} y + 2 - y^2\,dy$$

$$= \frac{1}{2}\left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right]\Big|_{-1}^{2}$$

$$= \frac{1}{2}\left\{\left(\frac{1}{2}(2)^2 + 2(2) - \frac{1}{3}(2^3)\right) - \left(\frac{1}{2}(-1)^2 + 2(-1) - \frac{1}{3}(-1)^3\right)\right\}$$

$$= \frac{1}{2}\left[\frac{10}{3} - \left(-\frac{7}{6}\right)\right]$$

$$= \boxed{\frac{9}{4}} = \boxed{2.25}$$

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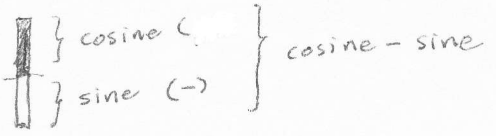
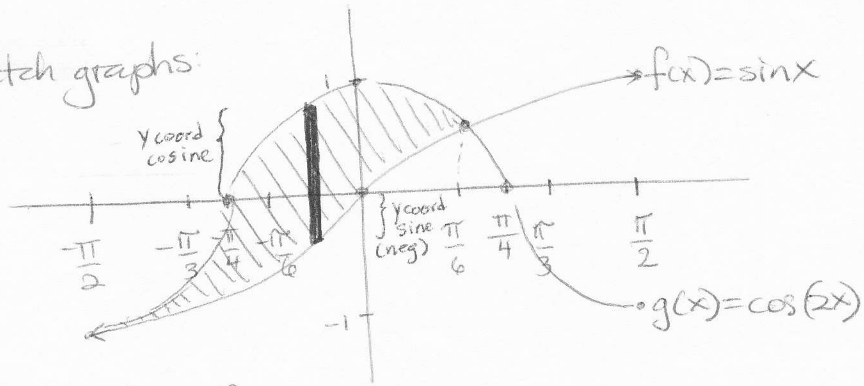


(4) $\begin{cases} f(x) = \sin x \\ g(x) = \cos 2x \end{cases} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{6}$

- sketch graphs:



$f(x) = \sin x$
$f(0) = (0)$
period $= 2\pi$

$g(x) = \cos 2x$
$g(0) = \cos(2 \cdot 0) = 1$
period $\frac{2x}{2} = \frac{2\pi}{2}$
$x = \pi$

- Find points of intersection

$\sin x = \cos(2x)$

$\sin x = 1 - 2\sin^2 x$ trig identity ⇒ write all terms using sines.

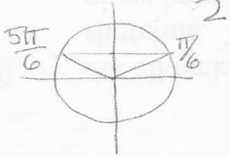
$2\sin^2 x + \sin x - 1 = 0$ Set $= 0$.

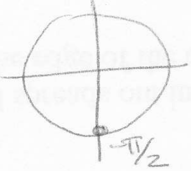
$(2\sin x - 1)(\sin x + 1) = 0$ factor

$2\sin x - 1 = 0$ $\sin x + 1 = 0$ set factors $= 0$

$2\sin x = 1$ $\sin x = -1$ solve resulting trig equations

$\sin x = \frac{1}{2}$ $x = -\frac{\pi}{2}$





$x = \frac{\pi}{6}$

- Identify uppermost function: $\cos(2x)$ and construct integral.

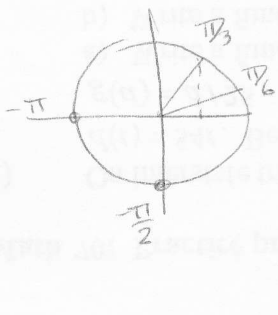
$$\text{Area} = \int_{-\pi/2}^{\pi/6} \cos(2x) - \sin x \, dx$$

$$= \int_{-\pi/2}^{\pi/6} \cos(2x)\,dx - \int_{-\pi/2}^{\pi/6} \sin x \, dx$$

$u = 2x$
$du = 2dx$
$u(-\pi/2) = 2(-\pi/2) = -\pi$
$u(\pi/6) = 2(\pi/6) = \pi/3$

$$= \frac{1}{2}\int_{-\pi}^{\pi/3} \cos u \, du - \int_{-\pi/2}^{\pi/6} \sin x \, dx$$

$$= \frac{1}{2}\Big[+\sin u\Big]_{u=-\pi}^{u=\frac{\pi}{3}} + \Big[\cos x\Big]_{x=-\pi/2}^{x=\pi/6}$$



$$= \frac{1}{2}\left(\sin\frac{\pi}{3} - \sin(-\pi)\right) + \left(\cos\frac{\pi}{6} - \cos\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 0\right) + \left(\frac{\sqrt{3}}{2} - 0\right)$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} = \boxed{\frac{3\sqrt{3}}{4}} \approx 1.299$$

1.299 GC check

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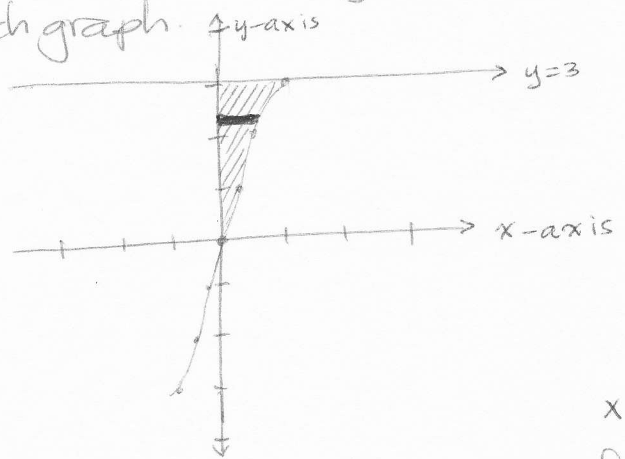
(5) $\begin{cases} f(y) = \dfrac{y}{\sqrt{16-y^2}} & \Rightarrow \text{only defined for } -4<y<4 \\ g(y) = 0 & \Rightarrow x=0 \text{ (vertical)} \\ y=3 & \Rightarrow y=3 \text{ (horizontal)} \end{cases}$

y	f(y)	(x,y)
-3	-1.134	≈(-1,-3)
-2	-0.577	≈(-.6,-2)
-1	-0.258	≈(-.3,-1)
0	0	≈(0,0)
1	0.258	≈(.3,1)
2	0.577	≈(.6,2)
3	1.134	≈(1,3)

use GC table and reverse x & y

• sketch graph



$x = \dfrac{y}{\sqrt{16-y^2}}$ $x=0$

• Determine rightmost function $f(y) = \dfrac{y}{\sqrt{16-y^2}}$ and leftmost $g(y)=0$

$$\int_{y=0}^{y=3} \left(\frac{y}{\sqrt{16-y^2}} - 0 \right) dy$$

do this in x?

$x = \dfrac{y}{\sqrt{16-y^2}}$ } can't isolate y ☹

Not possible.

$u = 16-y^2$

$du = -2y\,dy$

$u(0) = 16-0^2 = 16$ (lower)

$u(3) = 16-3^2 = 7$ (upper)

$$= -\frac{1}{2}\int_{16}^{7} u^{-1/2}\,du$$

$$= \frac{1}{2}\int_{7}^{16} u^{-1/2}\,du$$

$$= \frac{1}{2}\left[\frac{2}{1}u^{1/2}\right]\Big|_{7=u}^{16=u}$$

$$= u^{1/2}\Big|_{u=7}^{u=16}$$

$$= \sqrt{16} - \sqrt{7}$$

$$= \boxed{4-\sqrt{7}} \approx 1.354$$

GC check: 1.354 ✓

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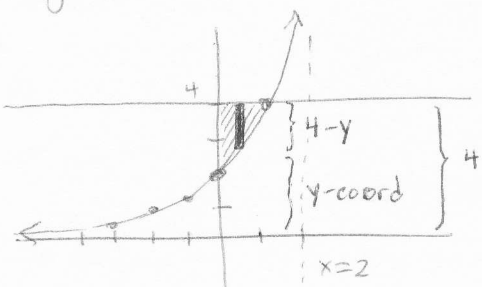
(6) $\begin{cases} g(x) = \dfrac{4}{2-x} & \Rightarrow \text{rational, asymptote at } x=2. \\ y=4 & \Rightarrow \text{horizontal} \\ x=0 & \Rightarrow \text{vertical} \end{cases}$

x	y
0	2
1	4
-1	4/3
-2	1
-3	4/5
-4	2/3

Use GC

- sketch graphs



- points of intersection (1,4). (see chart of points)
- Determine uppermost function $y=4$

$$\text{Area} = \int_0^1 \left(4 - \frac{4}{2-x}\right) dx$$

$$= 4\int_0^1 dx - 4\int_0^1 \frac{1}{2-x}\,dx$$

$u = 2-x$
$du = -dx$
$u(0) = 2-0 = 2$ lower
$u(1) = 2-1 = 1$ upper

$$= 4\int_0^1 dx + 4\int_2^1 \frac{1}{u}\,du$$

$$= 4x\Big|_0^1 + 4\ln|u|\Big|_2^1$$

$$= (4-0) + 4(\ln 1 - \ln 2)$$

$$= \boxed{4 - 4\ln 2} \approx 1.227$$

GCcheck: 1.227 ✓

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(7) $\begin{cases} f(x) = \sec\frac{\pi x}{4} \cdot \tan\frac{\pi x}{4} \\ g(x) = (\sqrt{2}-4)x + 4 \\ x = 0 \end{cases}$

$\Rightarrow$ Use GC; asymptotes where $\cos\left(\frac{\pi x}{4}\right) = 0$ $\quad \frac{\pi x}{4} = \frac{\pi}{2} \Rightarrow x = 2$

$\Rightarrow$ line, $m = \sqrt{2}-4$, $b = 4$

$\Rightarrow$ vertical

$\sqrt{2}-4 \approx -2.6$

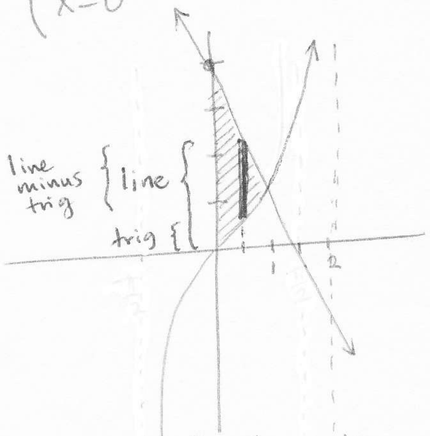
x-int (set $y = 0$)

$0 = (\sqrt{2}-4)x + 4$

$-4 = (\sqrt{2}-4)x$

$1.55 \approx \frac{-4}{\sqrt{2}-4} = x$

$\frac{\pi}{2} \approx 1.57$



- Find point of intersection: $\sec\frac{\pi x}{4} \cdot \tan\frac{\pi x}{4} = (\sqrt{2}-4)x + 4$

Use GC: ISECT/Intersect $\quad x = 1$

check

$\sec\left(\frac{\pi}{4}\right) \cdot \tan\frac{\pi}{4} = \sqrt{2} - 4 + 4$

$\sqrt{2} \cdot 1 = \sqrt{2}$ ✓

- Determine uppermost function

$g(x) = (\sqrt{2}-4)x + 4.$

$$\text{Area} = \int_0^1 \left[(\sqrt{2}-4)x + 4 - \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}\right] dx$$

$$= \int_0^1 \sqrt{2}x\,dx - \int_0^1 4x\,dx + \int_0^1 4\,dx - \int_0^1 \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}\,dx$$

$\begin{cases} u = \frac{\pi x}{4} \\ du = \frac{\pi}{4}dx \\ u(0) = \frac{\pi}{4}(0) = 0 \quad \text{lower} \\ u(1) = \frac{\pi}{4}(1) = \frac{\pi}{4}. \quad \text{upper} \end{cases}$

$$= \frac{\sqrt{2}}{2}x^2\Big|_0^1 - \frac{4}{2}x^2\Big|_0^1 + 4x\Big|_0^1 - \frac{4}{\pi}\int_0^{\pi/4} \sec u \tan u\,du$$

$$= \left(\frac{\sqrt{2}}{2} - 0\right) - (2-0) + (4-0) - \frac{4}{\pi}(\sec u)\Big|_0^{\pi/4}$$

$\frac{d}{dx}\sec x = \sec x \tan x$!

$$= \frac{\sqrt{2}}{2} - 2 + 4 - \frac{4}{\pi}\left(\sec\frac{\pi}{4} - \sec 0\right)$$

$$= \frac{\sqrt{2}}{2} + 2 - \frac{4}{\pi}(\sqrt{2} - 1)$$

$$= \boxed{\frac{\sqrt{2}}{2} + 2 - \frac{4\sqrt{2}}{\pi} + \frac{4}{\pi}} \approx 2.180$$

GC ≈ 2.180 ✓

Math 250 7.1

(8) Find the area of the region bounded by

$x_1 = y^2 \Rightarrow (x,y) = (y^2, y) = (x, \pm\sqrt{x})$ depends

$x_2 = 6 - y \Rightarrow (x,y) = (6-y, y) = (x, 6-x)$

$x = y^2$

$\pm\sqrt{x} = y$

$y_1 = \sqrt{x}$

$y_3 = -\sqrt{x}$

$x = 6 - y$

$x - 6 = -y$

$6 - x = y_2$

Points of intersection

$y^2 = 6 - y$

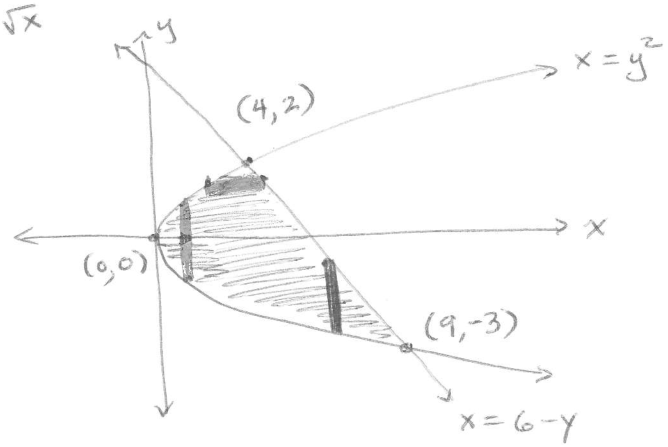
$y^2 + y - 6 = 0$

$(y+3)(y-2) = 0$

$y = -3 \qquad y = 2$

$x = 9 \qquad x = 4$

$(9,-3) \qquad (4,2)$



option 1: accumulate in x

$\xrightarrow{dx}$

y coord upper parabola minus y coord lower parabola + line y coord minus lower parabola y coord

$y_1 - y_3 \qquad y_2 - y_3$

Expressed in terms of x

Limits of integration in x.

dx.

$$\int_{x=0}^{x=4} \sqrt{x} - (-\sqrt{x})\,dx + \int_{x=4}^{x=9} 6 - x - (-\sqrt{x})\,dx$$

option 2: accumulate in y

$\uparrow dy$

x_1 parabola x coord

x_2 line x-coord

$x_2 - x_1$

Expressed in terms of y

Limits of integration in y

dy

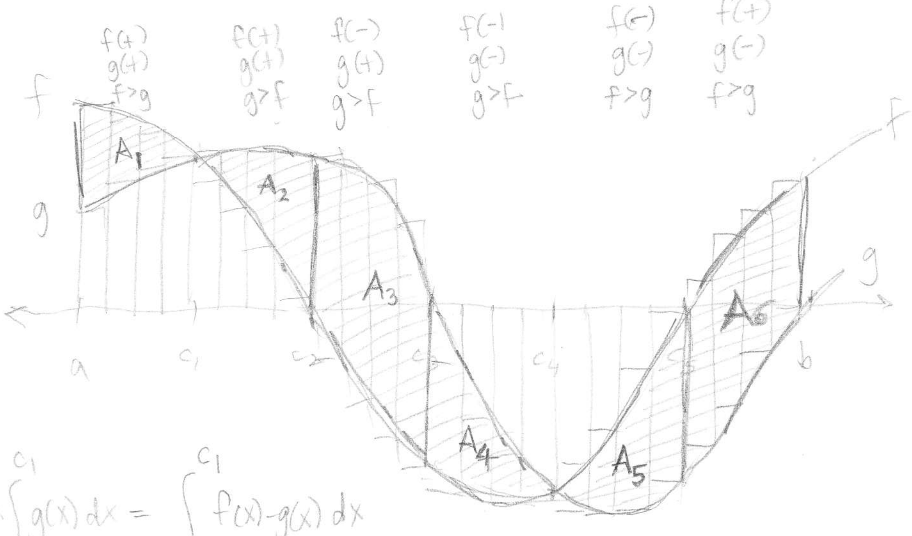
$$\int_{y=-3}^{y=2} 6 - y - y^2 \; dy$$

$$= \boxed{\frac{125}{6}}$$

M250

skip

GOAL: Find the shaded area.



f>g $A_1 = \int_a^{c_1} f(x)\,dx - \int_a^{c_1} g(x)\,dx = \int_a^{c_1} f(x)-g(x)\,dx$

g>f $A_2 = \int_{c_1}^{c_2} g(x)\,dx - \int_{c_1}^{c_2} f(x)\,dx = \int_{c_1}^{c_2} g(x)-f(x)\,dx$

g>f $A_3 = \int_{c_2}^{c_3} g(x)\,dx + \int_{c_2}^{c_3} -f(x)\,dx = \int_{c_2}^{c_3} g(x)-f(x)\,dx$

g>f $A_4 = -\int_{c_3}^{c_4} -g(x)\,dx + \int_{c_3}^{c_4} -f(x)\,dx = \int_{c_3}^{c_4} g(x)-f(x)\,dx$

f>g $A_5 = -\int_{c_4}^{c_5} -f(x)\,dx + \int_{c_4}^{c_5} -g(x)\,dx = \int_{c_4}^{c_5} f(x)-g(x)\,dx$

f>g $A_6 = \int_{c_5}^{b} f(x)\,dx + \int_{c_5}^{b} -g(x)\,dx = \int_{c_5}^{b} f(x)-g(x)\,dx$

In the end, it doesn't matter if f>0 or g>0
it only matters if f>g or g>f. So c_2, c_3 and c_5 are irrelevant

$$\text{Total } A = \underbrace{\int_a^{c_1} f(x)-g(x)\,dx}_{A_1} + \underbrace{\int_{c_1}^{c_4} g(x)-f(x)\,dx}_{(A_2+A_3+A_4)} + \underbrace{\int_{c_4}^{b} f(x)-g(x)\,dx}_{(A_5+A_6)}$$

We need to know
1. values c_i where $f(x)=g(x)$
2. between values c_i, is $f(x)>g(x)$ or $g(x)>f(x)$?